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Multiphoton ionization of a quantum well

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Abstract. The paper deals with two problems of the ionization of a single quantum well formed in a heterostructure.

In the first problem, the ionization is caused by an electromagnetic wave which has so low a frequency that the number of electromagnetic quanta necessary for the ionization is much more than unity. The electric field of the wave is oriented along the normal to the heterostructure boundaries, and is assumed to be uniform on the scale of the well width. Analytical expressions have been obtained to determine the multiphoton ionization probability of a rectangular well per unit of time. It has been shown that plots of ionization probability versus well width depend dramatically on whether the angular frequency of the field is larger or smaller than the inverse characteristic time of the electron motion in the classically forbidden region. In the former case, the ionization rate as a function of the well width at a fixed frequency (or as a function of the field frequency at a fixed width) is characterized by two types of peculiarity. We discuss the nature of these peculiarities.

The second problem is devoted to the ionization of a quantum well by an alternating electric field superimposed on a strong dc electric field. This problem is investigated analytically for a field frequency smaller than the inverse characteristic time of the electron motion through a potential barrier. The dependences of the ionization probability on the well width and the electric field amplitude are studied.

1. Introduction

The multiphoton process of ionization of atoms in strong electromagnetic fields has attracted the attention of investigators for many years (see, for example, [1–6] and references therein). The perturbation theory is not appropriate for calculating the ionization rate in this case. Keldysh was the first to solve such a problem for atoms radiated by a strong laser field, as early as 1964 [1]. He demonstrated that the frequency dependence of the multiphoton ionization probability is determined by the parameter

$$\gamma = \frac{\omega}{\omega_t} = \frac{\omega \hbar \kappa}{|e\mathcal{E}|} \tag{1}$$

where ω and $\mathcal{E}$ are the angular frequency and the electric field amplitude of the wave, respectively, $\hbar$ is Planck's constant, e is the electron charge, and κ^{-1} is the space-scale characterizing the exponential decrease of the electron wave function at large distances from the atomic nucleus. The quantity ω_t^{-1} can be considered as the characteristic time of the electron motion in the classically forbidden region in the presence of the dc field $\mathcal{E}$. At low frequencies ($\gamma \ll 1$), the ionization is reduced to the tunnelling of an electron through a slowly time-varying potential barrier. At high frequencies ($\gamma \gg 1$), the ionization can be interpreted as the absorption of a large number of quanta. In the latter case, the probability of absorbing n quanta is proportional to $1/\gamma^{2n}$, i.e. it decreases abruptly with increasing n (n is assumed to be larger than some threshold value). The Keldysh approach implies that an alternating field does not practically affect the atomic ground state, but influences essentially the free-electron motion. This motion is described by the function that is an exact solution of the Schrödinger equation for an electron in a uniform alternating electric field, rather than by a plane wave.

In some previous papers [2, 5, 6] one-dimensional models of atoms were treated in order to understand the complicated nature of the ionization process. Meanwhile, there exist natural objects whose ionization can be described by a one-dimensional model. These are quantum wells in semiconductor heterostructures. Up to now the intersubband electron transitions in quantum wells and the ionization processes due to a weak electromagnetic field have been considered, if one photon is involved in the process and, therefore, only the first order of the perturbation theory on the field amplitude is applicable [7, 8]. Below we shall focus on the ionization of a quantum well formed in a heterostructure and affected by a strong electromagnetic wave. The wave frequency ω is supposed to be so low that the quantum energy $\hbar\omega$ is smaller than the depth of the energy level of an electron in the quantum well. The electric field of the wave is oriented along the normal to the heterostructure boundaries. The field is assumed to be uniform which corresponds to the much larger electromagnetic wavelength as compared to the width of the quantum well. We shall investigate the ionization rate as a function of the well width and the wave frequency, and point out the situation where the ionization is suppressed due to the interference of electron waves released from the well. This ionization suppression has not been discussed in previous papers.

Further on, we deal with the problem of the quantum well ionization by an alternating electric field $\boldsymbol{E}(t)$ in the presence of a strong dc electric field $\boldsymbol{E}_0$ parallel to $\boldsymbol{E}(t)$. This problem is urgent now, since, in particular, in photodetectors with quantum wells a heterostructure exists simultaneously in the dc ‘pulling’ field and in the wave field [7].

We suppose the heterostructure temperature to be so low that two conditions hold: (i) all electrons occupy the lowest energy level, and the states of the continuous spectrum are free; (ii) the characteristic time between collisions of an electron with thermal phonons is much larger than the characteristic time of the electron motion in the classically forbidden region. The latter condition permits us to neglect the electron–phonon scattering and to describe the behaviour of an electron by a Schrödinger equation.

In section 2 we shall derive the analytical expression for the ionization probability per unit of time, when a rectangular quantum well is subjected to a uniform ac electric field. In section 3 we shall analyse the expressions obtained in two limit cases (low frequency and high frequency) and give numerical results. In section 4 we shall obtain the analytical expression for the ionization probability of a rectangular quantum well placed in dc and ac electric fields simultaneously.

2. The ionization probability

Let us consider the one-dimensional motion of an electron in a potential well in the presence of an ac electric field $\mathcal{E}\cos\omega t$ turned on at $t = 0$ and directed along the x axis. The electron wave function $\Psi(x, t)$ obeys the equation

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x) - e\mathcal{E}x\cos\omega t\right)\Psi \qquad (2)$$

with the initial condition

$$\Psi(x,0) = \Psi_0(x)$$

where m is the effective mass of the electron, $U(x)$ is the potential energy of the well, and $\Psi_0(x)$ is the wave function of an electron stationary state in the quantum well. In this paper we assume that m is independent of x and that $e\mathcal{E} > 0$. We restrict ourselves to the case of the ground initial state of an electron in a rectangular well U_0 deep and a wide. It is well known that in this case

$$\Psi_0(x) = \begin{cases} C_0\cos(kx) & |x| < \frac{1}{2}a \\ C_0\cos\left(\frac{1}{2}ka\right)\exp\left[-\kappa\left(|x| - \frac{1}{2}a\right)\right] & |x| > \frac{1}{2}a \end{cases} \tag{3}$$

where

$$C_0 = \frac{\sqrt{\kappa}}{\sqrt{1+\frac{1}{2}\kappa a}} \qquad k = \frac{1}{\hbar}\sqrt{2m(U_0 - |E_0|)} \qquad \kappa = \frac{1}{\hbar}\sqrt{2m|E_0|}. \tag{4}$$

$E_0 < 0$ is the energy of the initial stationary state in the well. Everywhere below we suppose that

$$\hbar\omega \ll |E_0|. \tag{5}$$

In order to find the wave function it is convenient to rewrite equation (2) in the integral form and then to replace the wave function in the integral term by the unperturbed function

$$\Psi_0(x)\exp\left(-\frac{\mathrm{i}}{\hbar}E_0 t\right)$$

(see appendix A). This approximate procedure suggested in [2] implies that the condition $e\mathcal{E}a \ll |E_0|$ is satisfied and the overall loss of carriers is negligible over the time t of our consideration. As has been shown in appendices A and B, for large positive x, t we have

$$\Psi(x,t) \simeq \frac{\mathrm{i}U_0 m}{\hbar}\sqrt{\frac{\kappa}{1+\frac{1}{2}\kappa a}}\sum_{n\geqslant\nu}^{\infty}\frac{1}{p_n}f_n(p_n)\exp\left\{\frac{\mathrm{i}}{\hbar}\left[\left(p_n + \frac{e\mathcal{E}}{\omega}\sin\omega t\right)x \right.\right.$$
$$\left.\left. +(|E_0| - n\hbar\omega)t + \frac{p_n e\mathcal{E}}{m\omega^2}\cos\omega t + \frac{e^2\mathcal{E}^2}{8m\omega^3}\sin 2\omega t\right]\right\}. \tag{6}$$

Here

$$p_n = \sqrt{2m\hbar\omega(n-\nu)} \qquad (n \geqslant \nu) \tag{7a}$$

$$\nu \equiv \frac{|E_0|}{\hbar\omega}\left(1 + \frac{1}{2\gamma^2}\right) \tag{7b}$$

$$f_n(p_n) \simeq \frac{1}{2}\left[\frac{\sin(\mathrm{i}\kappa + k)a/2}{\mathrm{i}\kappa + k} + \frac{\sin(\mathrm{i}\kappa - k)a/2}{\mathrm{i}\kappa - k}\right]\sqrt{\frac{\hbar\omega\gamma}{\pi|E_0|\sqrt{1+\gamma^2}}}$$
$$\times\exp\left[-\frac{|E_0|}{\hbar\omega}\left[f(\gamma) + \frac{p_n^2}{\hbar^2\kappa^2}(Ar\sinh\gamma - \frac{\gamma}{\sqrt{1+\gamma^2}})\right]\right]$$
$$\times\left[\exp\left(-\frac{\mathrm{i}p_n\kappa}{\omega m\gamma}\sqrt{1+\gamma^2}\right) + (-1)^n\exp\left(\frac{\mathrm{i}p_n\kappa}{\omega m\gamma}\sqrt{1+\gamma^2}\right)\right] \tag{8}$$

where

$$f(\gamma) = \left(1 + \frac{1}{2\gamma^2}\right)Ar\sinh\gamma - \frac{\sqrt{1+\gamma^2}}{2\gamma} \tag{9}$$

is the function obtained by Keldysh [1] for the first time. The parameter γ is given by equation (1) in which κ is the quantity defined by equation (4). The quantity p_n is the momentum (averaged over the field period) of the electron released from the well after absorbing n field quanta. The number ν determines the minimum electromagnetic quantum number required for the ionization. Equation (6) is valid if the conditions

$$x \gg \frac{a}{2}, \frac{e\mathcal{E}}{m\omega^2}, \sqrt{\frac{\hbar}{2m\omega}} \tag{10a}$$

$$x\sqrt{\frac{m}{2\hbar\omega}} \ll t \ll w^{-1} \tag{10b}$$

are satisfied where w is the ionization probability per unit of time (see equations (15a) and (15b)). Conditions (10a) and (10b) do not contradict one another if, at least, the inequality

$$w \ll \omega \tag{10c}$$

holds. This inequality is natural because one expects that $t > 1/\omega$; hence the condition $wt \ll 1$ means also that $w \ll \omega$. The condition of validity for equation (8) is given by equation (5) if $\gamma \gtrsim 1$, and by the inequality

$$\sqrt{\frac{e\mathcal{E}}{\kappa|E_0|}} \ll 1 \qquad \text{if } \gamma \ll 1 \tag{11}$$

(see appendix B).

By using function (6) one can readily calculate the electron current at large positive x and t:

$$j(x,t) = -\frac{\mathrm{i}\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right). \tag{12}$$

The x-independent part of this current, averaged over the period of the electromagnetic field, is equal to

$$j_+ \simeq \frac{U_0^2 m\kappa}{\hbar^2(1+\frac{1}{2}\kappa a)}\sum_{n\geqslant\nu}^{\infty}\frac{1}{p_n}|f_n(p_n)|^2. \tag{13}$$

Obviously, the total electron current from the well is $2j_+$. This current determines the ionization probability w per unit of time, namely

$$w = 2j_+.$$

Each term in equation (13), multiplied by two, is the rate of the n-quantum ionization. It should be noted that the averaging procedure used for deriving equation (13) is relevant because, according to inequality (10c), the characteristic time of the ionization is much larger than the alternating-field period.

Substituting expression (8) into equation (13) and taking into account the fact that according to the transcendental equation governing the electron spectrum in the quantum well, the following relation holds:

$$\left[\frac{\sin(\mathrm{i}\kappa+k)a/2}{\mathrm{i}\kappa+k} + \frac{\sin(\mathrm{i}\kappa-k)a/2}{\mathrm{i}\kappa-k}\right]^2 = \frac{2\hbar^2\mathrm{e}^{\kappa a}}{mU_0^3}(U_0-|E_0|)|E_0| \tag{14}$$

we can represent the ionization probability as

$$w = \frac{\mathrm{e}^{\kappa a}}{1+\frac{1}{2}\kappa a}\left(1-\frac{|E_0|}{U_0}\right)w_\delta \tag{15a}$$

$$w_\delta \simeq 2\frac{|E_0|}{\hbar}\sqrt{\frac{3e\mathcal{E}}{2\pi\kappa|E_0|}}\,\mathrm{e}^{-2(|E_0|/\hbar\omega)f(\gamma)}\sqrt{\frac{2\gamma^3}{3\pi(1+\gamma^2)}}$$
$$\times\sum_{n\geqslant\nu}^{\infty}\frac{\exp\left(-2\left(Ar\sinh\gamma-\gamma/\sqrt{1+\gamma^2}\right)(n-\nu)\right)}{\sqrt{n-\nu}}$$
$$\times\left[1+(-1)^n\cos\left(4\sqrt{\frac{|E_0|}{\hbar\omega}\left(1+\frac{1}{\gamma^2}\right)(n-\nu)}\right)\right]. \qquad (15b)$$

The quantity w_δ is the rate of ionizing the well with $U(x)$ proportional to $\delta(x)$ (see [2]). It should be emphasized that formulas (8) and (15*b*) are valid if the inequality $\hbar\omega \ll |E_0|$ is satisfied. This inequality contradicts the condition $\hbar\omega \geqslant |E_0|$ for the one-quantum ionization. Consequently, equation (15*b*) does not describe the rate of the one-quantum ionization in a weak electromagnetic field.

As we have mentioned in section 1, the characteristic time τ between collisions of an electron with thermal phonons should be much larger than the characteristic time τ_f of the electron motion in the classically forbidden region. In the quasiclassical approximation the latter quantity can be estimated as

$$\tau_f \sim \frac{1}{\omega}\ln(\sqrt{1+\gamma^2}+\gamma).$$

Thus, equations (15*a*) and (15*b*) are relevant at temperatures so low that the condition

$$\tau\omega \gg \ln(\sqrt{1+\gamma^2}+\gamma) \qquad (16)$$

holds. This condition evolves as $\tau\omega_t \gg 1$, if $\gamma \ll 1$, and as $\tau\omega \gg \ln(2\gamma)$, if $\gamma \gg 1$.

3. Discussion and numerical results

Let us analyse formulas (15*a*) and (15*b*). In the high-frequency limit ($\gamma \gg 1$), the terms of the sum in equation (15*b*) decrease abruptly if n increases. Therefore, it is sufficient to employ only the largest term, the number n of which is the closest one to ν (let us designate this number as n_{min}). As a result, we have

$$w \simeq \frac{\mathrm{e}^{\kappa a}}{1+\frac{1}{2}\kappa a}\left(1-\frac{|E_0|}{U_0}\right)2\frac{\sqrt{\hbar\omega|E_0|}}{\pi\hbar}$$
$$\times\left(\frac{e\mathcal{E}}{2\hbar\omega\kappa}\right)^{2n_{min}}\exp\left(2n_{min}-\frac{|E_0|}{\hbar\omega}\right)\frac{1}{\sqrt{n_{min}-|E_0|/\hbar\omega}}$$
$$\times\left[1+(-1)^{n_{min}}\cos\left(4\sqrt{\frac{|E_0|}{\hbar\omega}}\sqrt{n_{min}-\frac{|E_0|}{\hbar\omega}}\right)\right]. \qquad (17)$$

According to the definition of n_{min} one has $n_{min}-1 < |E_0|/\hbar\omega \leqslant n_{min}$. The situation where $|E_0| = n_{min}\hbar\omega$ corresponds to the threshold of the n_{min}-quantum absorption.

The ionization rate in the high-frequency limit has some peculiarities. Let the well depth, the ac-field amplitude and frequency be fixed. If the well width increases, the quantity $|E_0|$ increases, too. As a consequence, the ratio $|E_0|/\hbar\omega$ tends to n_{min}, and the ionization probability becomes proportional to $(n_{min}-|E_0|/\hbar\omega)^{-1/2} \to \infty$ if n_{min} is even, and proportional to $(n_{min}-|E_0|/\hbar\omega)^{1/2} \to 0$ if n_{min} is odd. Thus, if a threshold corresponds to even n_{min}, the ionization probability is infinite. And if a threshold corresponds to odd n_{min}, this probability approaches zero. The singularity of the ionization rate at even n_{min} is connected with the singularity of the density of electron states in the one-dimensional case. For odd n_{min} the ionization rate singularity is suppressed. We interpret this fact as follows. If an electron absorbs a field quantum, the electron wave function changes its parity. Both the ground state in a rectangular well and the lowest state of the continuous spectrum are even; therefore, a transition between these two states is possible if the electron absorbs an even number of the field quanta. The transitions accompanied by the absorption of an odd number of quanta are forbidden. Hence, if the well width increases, the ionization probability becomes equal in turn to infinity and zero at the thresholds.

Besides this, there exists a second type of peculiarity of the ionization probability as a function of the well width. According to equation (17), the ionization probability equals zero when the condition

$$4\sqrt{\frac{|E_0|}{\hbar\omega}}\sqrt{n_{min}-\frac{|E_0|}{\hbar\omega}} = \begin{cases} \pi(2k-1) & k=1,2,\ldots \text{ if } n_{min} \text{ is even} \\ 2\pi k & k=1,2,\ldots \text{ if } n_{min} \text{ is odd} \end{cases} \tag{18}$$

is satisfied. Formally, under this condition the matrix element connecting the ground state and the free state vanishes. The physical reason for this effect can be interpreted as follows. The exponential functions in the last pair of square brackets in equation (8) are of the form $\exp(\pm(\mathrm{i}/\hbar)p_n x_0)$ where $x_0 \simeq \hbar\kappa/m\omega$. Hence, in accordance with equations (6) and (8), the wave-function component with the momentum p_n can be treated as a superposition of two waves outgoing from two different points whose distance apart equals $2x_0$. If the ratio of the distance $2x_0$ and the electron wavelength $2\pi\hbar/p_n$ is an integer at odd n_{min} or a half-integer at even n_{min} (these conditions are expressed by equation (18)), the interference of two partial waves leads to zero total amplitude of the electron wave with the momentum p_n. Thus, the second type of peculiarity of the ionization rate is connected with the interference of two electron waves released from the well.

Strictly speaking, because of the restriction $w \ll \omega$ we cannot state that, for some parameters of the well and the field, the ionization rate tends to infinity. We also cannot state that the ionization probability described by equation (17) is exactly equal to zero because, in contrast to equations (15*a*) and (15*b*), formula (17) does not take into account the n-quantum ionization processes with $n > n_{min}$. Furthermore, electron energy levels in a semiconductor have a finite width due to collisions. One can estimate the minimum value of the quantity $(n_{min}-\nu)^{1/2}$ as $(\omega\tau)^{-1/2}$ where τ is the characteristic time of the electron motion between collisions. According to inequality (16), the product $\omega\tau$ is very large, and one can expect that the singularities of the ionization rate will manifest themselves.

Let us turn to the low-frequency limit ($\gamma \ll 1$). First of all, we separate the term with n_{min} from the sum in equation (15*b*) because this term can be singular. In the remaining sum we rewrite the exponential function as $\exp(-\frac{2}{3}\gamma^3(n-\nu))$. This is a slow function of n for $\gamma \ll 1$. In contrast, the second term in the square brackets in equation (15*b*) oscillates quickly if n changes, and we can omit this term when we calculate the sum over n. After that, the summation from $n = n_{min}+1$ to $n=\infty$ can be replaced by the integration over the variable $\xi = \frac{2}{3}\gamma^3(n-\nu)$. As a result, we have

$$w \simeq \frac{\mathrm{e}^{\kappa a}}{1+\frac{1}{2}\kappa a}\left(1-\frac{|E_0|}{U_0}\right)2\frac{|E_0|}{\hbar}\sqrt{\frac{3e\mathcal{E}}{2\pi\kappa|E_0|}}\exp\left(-\frac{4}{3}\frac{\kappa|E_0|}{e\mathcal{E}}\right) \times\left\{1+\sqrt{\frac{2\gamma^3}{3\pi(n_{min}-\nu)}}\left[1+(-1)^{n_{min}}\cos\left(4\sqrt{\frac{|E_0|}{\hbar\omega\gamma^2}(n_{min}-\nu)}\right)\right]\right\}. \tag{19}$$

According to equation (19), the ionization rate as a function of the well width or the field frequency is singular at the points where $\nu = n_{min}$ for even n_{min}. The singularities are connected with the second term in the braces. Since $\gamma \ll 1$, this term becomes crucial if ν is very close to n_{min}. If we take into account a finite width of the electron energy levels, the minimum value of the quantity $(n_{min} - \nu)^{1/2}$ can be estimated as $(\omega\tau)^{-1/2}$ where τ is the characteristic time of the electron motion between collisions, introduced above. Therefore, the maximum value of the singular term in the braces is of the order of

$$\sqrt{\frac{\gamma^3}{(n_{min} - \nu)}} \sim (\gamma^3 \omega\tau)^{1/2}.$$

For sufficiently small γ, this quantity is much less than unity, and below we shall neglect the second term in the braces. In this approximation the ionization rate does not depend on ω.

In equation (19) the factor

$$\exp\left(-\frac{4}{3}\frac{\kappa|E_0|}{e\mathcal{E}} + \kappa a\right)$$

is none other than the quasiclassical transparency of the triangular barrier formed by the well potential and the dc electric field $\mathcal{E}$. This transparency is much smaller than unity (see inequality (11)). The factor

$$\sqrt{\frac{1}{2\pi\kappa|E_0|}3e\mathcal{E}}$$

determines the part of the ac-field period where the barrier width is minimum and the tunnelling is proceeding most effectively. This becomes evident if we compare the ionization rate in the low-frequency limit (see equation (19)) and the ionization rate in the dc field of magnitude equal to the ac-field amplitude (see equation (22) and the paragraph after it). It is interesting that, on the one hand, equation (19) describes the electron tunnelling from the well through the barrier varying in time. On the other hand, this equation has been derived from equation (15*b*), according to which the ionization in the low-frequency limit is an aggregate of a great number of multiphoton processes involving $n_{min}, n_{min} + 1, n_{min} + 2, \ldots, n_{min} + A/\gamma^3$ photons, respectively, where A is a constant of the order of unity. The latter statement is a consequence of the fact that the contribution of the n-photon process to the ionization rate is proportional to $\exp(-\frac{2}{3}\gamma^3(n - \nu))$ (see the paragraph above equation (19)).

We emphasize that equation (19) is valid in the frequency range $w \ll \omega \ll e\mathcal{E}/\hbar\kappa$. The upper limit is defined by the condition $\gamma \ll 1$. The left-hand inequality means that the carrier population in the well does not change in essence during the field period.

Up to now we considered the electron transitions from the ground state of a quantum well to the states of the continuous spectrum. If there are several levels in a well, it is possible to obtain the probability of multiphoton ionization of the Nth quantum state ($N = 0$ corresponds to the ground state). It can be shown that for even N, the ionization probability is described by equations (15*a*) and (15*b*), in which $|E_0|$, κ, γ, ν should be replaced by $|E_N|$:

$$\kappa_N = \frac{1}{\hbar}\sqrt{2m|E_N|} tqs \gamma_N = \frac{1}{e\mathcal{E}}\hbar\omega\kappa_N \qquad \nu_N = \frac{|E_N|}{\hbar\omega}\left(1 + \frac{1}{2\gamma_N^2}\right)$$

respectively. If N is odd, in addition to the replacements mentioned above, it is necessary to substitute $(-1)^{n+1}$ instead of $(-1)^n$ in equation (15*b*). It is necessary to have in mind that, when there are several levels in the well, our formulas for the ionization probability are valid, if the transitions between the levels in the well are not resonant, i.e. the interval between the levels, divided by $\hbar\omega$, is not an integer.

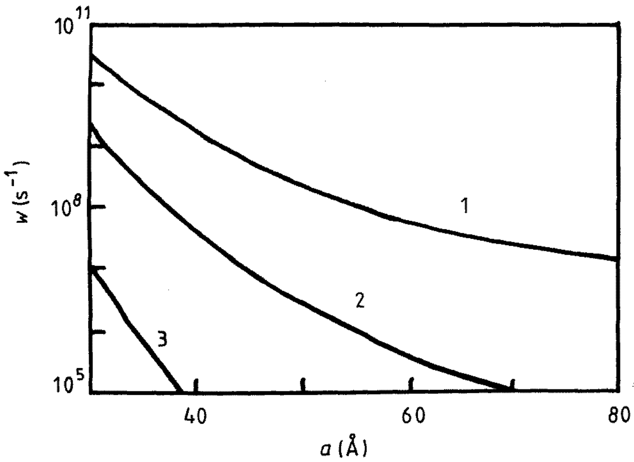

Figure 1. The ionization probability in the low-frequency limit versus the width of a well 0.3 eV deep. The amplitude of the alternating electric field equals 130 kV cm^{-1} (curve 1), 100 kV cm^{-1} (curve 2), 70 kV cm^{-1} (curve 3).

The expressions derived above are rather bulky. The explicit dependences of the ionization probability on the well width and on the field amplitude and frequency are plotted using equations (17) and (19) (without the second term in the braces in the latter equation). We choose the parameters $U_0 = 0.3$ eV and $m = 0.07m_0$ (m_0 is the free-electron mass) which approximately correspond to the n-type $Al_xGa_{1-x}As/GaAs/Al_xGa_{1-x}As$ heterostructures. Figure 1 shows the dependence $w(a)$ in the low-frequency limit ($\gamma \ll 1$), when the ionization can be considered as the tunnelling through the slowly time-varying barrier. In this case, the ionization probability does not explicitly depend on the frequency, but the latter has to be much higher than w . The electric field amplitude is 130 kV cm^{-1} (curve 1), 100 kV cm^{-1} (curve 2), and 70 kV cm^{-1} (curve 3). The dependence of the ionization probability on the well width is a monotonic function. This is due to the fact that the depth of the energy level and the scale κ^{-1} of the wave-function localization are smooth functions of a. In the low-frequency limit, the largest values of the ionization probability correspond to narrow wells, the barrier widths of which are comparatively small because of the small depth of the energy level.

Figure 2 illustrates the dependence $w(a)$ in the high-frequency limit ($\gamma \gg 1$). In the calculations $\mathcal{E}$ is taken to be equal to 70 kV cm^{-1}, while $\omega/2\pi = 14$ THz. In this case, the dependence $w(a)$ is sharply nonmonotonic. Two peculiarities at the points $a \simeq 33$ Å, $a \simeq 52$ Å correspond to thresholds. The minimum number of quanta required for the ionization changes from three to four at the point $a \simeq 33$ Å, and from four to five at the point $a \simeq 52$ Å. If the well width tends to 33 Å from below, the minimum number of quanta, n_{min}, is equal to three; therefore, in accordance with equation (17), the threshold value of the ionization probability is zero. In contrast, at the point $a \simeq 52$ Å the threshold value of this probability tends to infinity. Since we use a logarithmic scale, in figure 2 we see two weak singularities. The peculiarity in the vicinity of the point $a = 48$ Å is of

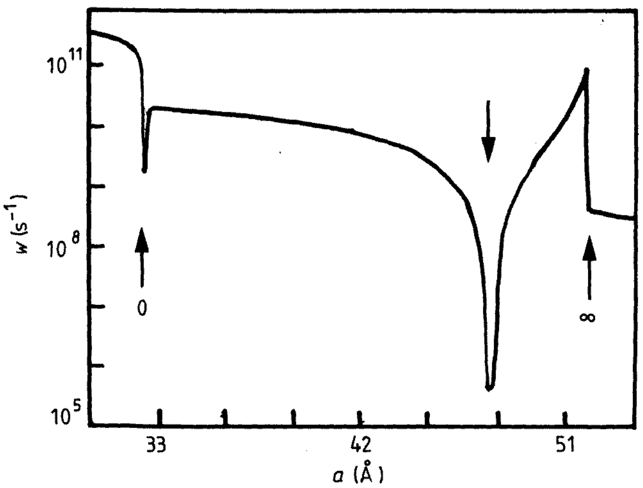

Figure 2. The ionization probability in the high-frequency limit versus the width of a well 0.3 eV deep. The electric field amplitude and frequency $\omega/2\pi$ are 70 kV cm^{-1} and 14 THz, respectively. The arrows with symbols 0 or ∞ correspond to the widths at which the ionization probability tends to 0 or ∞, respectively, and at which the minimum number of quanta required for the ionization changes by unity. The arrow above the curve corresponds to the width at which the ionization probability tends to zero because of the interference of two electron waves released from the well.

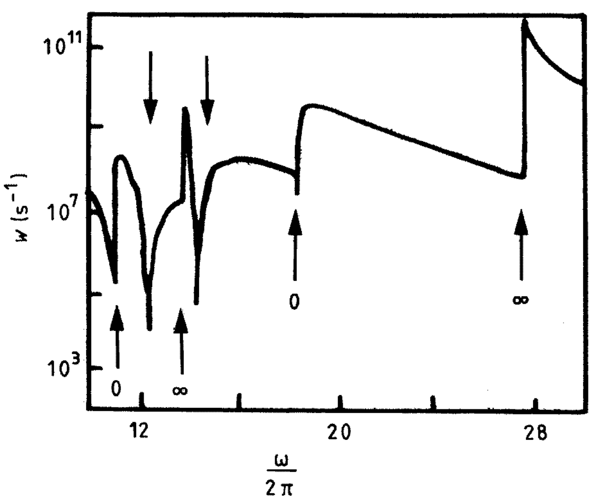

Figure 3. The ionization probability in the high-frequency limit versus the frequency of an electric field. The well is 0.3 eV deep and 50 Å wide. The field amplitude is 70 kV cm^{-1}. The arrows with symbols 0 or ∞ correspond to the frequencies at which the ionization probability tends to 0 or ∞, respectively, and at which the minimum number of quanta required for the ionization changes by unity. The arrows above the curve correspond to the frequencies at which the ionization probability tends to zero because of the interference of two electron waves released from the well.

different nature. The value $a \simeq 48$ Å is a root of equation (18) at $n_{min} = 4$. This type of peculiarity has been named above as the second type and is due to the interference of two electron waves released from the well. The dependence of the ionization rate on the field frequency in the limit $\gamma \gg 1$ is depicted in figure 3. The effective electron mass and the well depth are the same as in figures 1 and 2. The well width equals 50 Å; the field amplitude is 70 kV cm^{-1}. The arrows under the curve point out the peculiarities corresponding to thresholds of the absorption. Symbols 0 or ∞ near an arrow mean that at this point the ionization rate tends to zero or infinity, respectively. These arrows (from the right to the left) correspond to the frequencies at which the minimum quantum number required for the ionization changes from 2 to 3, from 3 to 4, and so on. The arrows above the curve correspond to the points where the ionization is suppressed due to the interference of the electron waves.

In order to evaluate the efficiency of the multiphoton ionization, let us suppose that the electron concentration in the well is $n_s = 10^{12}$ cm^{-2}. For the parameters $U_0 = 0.3$ eV, $m = 0.07m_0$, $a \simeq 40$ Å, $\mathcal{E} = 70$ kV cm^{-1}, $\omega/2\pi = 14$ THz we have $w \simeq 10^{10}$ s^{-1} (see figure 2). Consequently, the number of electrons outgoing from the well per unit of time equals $n_s w \simeq 10^{22}$ s^{-1} cm^{-1}. The number of electromagnetic quanta incident upon a unit of the sample surface per unit of time is of the order of

$$N_{ph} \sim \frac{1}{8\pi\hbar\omega} c\sqrt{\epsilon}\mathcal{E}^2$$

where c is the light velocity, and ϵ is the dielectric permeability of the semiconductor. In our example we find $N_{ph} \sim 2.6 \times 10^{27}$ s^{-1} cm^{-1} if $\epsilon = 12.5$. Thus, the efficiency $Q = n_s w/N_{ph}$ is approximately equal to 3.8×10^{-6}. If in our example we vary only the electric field amplitude $\mathcal{E}$, the efficiency Q will be described by the formula

$$Q \sim 3.8 \times 10^{-6} \left(\frac{\mathcal{E}}{70}\right)^6$$

where $\mathcal{E}$ is expressed in kV cm^{-1}.

We have considered two limit cases—the low-frequency and the high-frequency one. The typical crossover ω_{cr} between low and high frequencies is determined by the condition $\gamma \sim 1$, i.e.

$$\omega_{cr} \sim \frac{e\mathcal{E}}{\sqrt{2m|E_0|}}.$$

If we suppose the electron effective mass to equal $0.07m_0$, the depth and the width of a rectangular well to be 0.3 eV and 50 Å, respectively, we can find that the ground-state energy in the well is $E_0 \simeq -0.21$ eV. Therefore, for $\mathcal{E} = 70$ kV cm^{-1} we have $\omega_{cr}/2\pi \sim 2.8$ THz.

4. Ionization of a quantum well by dc and ac electric fields

Now let us consider the ionization processes, when a quantum well is placed in dc and ac uniform electric fields simultaneously. The fields are supposed to be parallel and directed along the x axis. We restrict ourselves to the case of sufficiently low frequencies, where the characteristic time of the electron motion in the classically forbidden region is much less than the period of an ac electromagnetic field. First we calculate the ionization probability in the dc field $\mathcal{E}_0$. This problem was investigated numerically in [9], and we perform an analytical solution. The wave function of an electron tunnelling from a rectangular potential well in the dc field $\mathcal{E}_0$ is found in appendix C. For large positive x, t we have ($e\mathcal{E}_0$ is assumed to be positive)

$$\Psi(x,t) \simeq \frac{U_0 m^{3/4}}{\hbar^{3/2}\sqrt[4]{2e\mathcal{E}_0}\sqrt{1+\frac{1}{2}\kappa a}}\left[\frac{\sin(\mathrm{i}\kappa+k)a/2}{\mathrm{i}\kappa+k}+\frac{\sin(\mathrm{i}\kappa-k)a/2}{\mathrm{i}\kappa-k}\right]\mathrm{e}^{-2|E_0|\kappa/3e\mathcal{E}_0}$$
$$\times\frac{1}{\sqrt[4]{x-(1/e\mathcal{E}_0)|E_0|}}\exp\left\{\mathrm{i}\frac{\pi}{4}-\mathrm{i}\frac{E_0}{\hbar}t+\mathrm{i}\frac{2}{3\hbar}\sqrt{2me\mathcal{E}_0}\left(x-\frac{|E_0|}{e\mathcal{E}_0}\right)^{3/2}\right\}. \tag{20}$$

The conditions of validity of this equation are

$$e\mathcal{E}_0 a \ll |E_0| \qquad wt \ll 1 \tag{21a}$$

$$t \gg \sqrt{\frac{2m}{e\mathcal{E}_0\kappa}} \qquad \sqrt{\frac{e\mathcal{E}_0}{|E_0|\kappa}} \ll 1 \tag{21b}$$

$$\frac{\sqrt{\hbar}}{\sqrt[4]{2me\mathcal{E}_0}(x-(1/e\mathcal{E}_0)|E_0|)^{3/4}} \ll 1 \tag{21c}$$

where w is the ionization probability per unit of time (see equation (22)). Conditions (21a) and (21b) for t do not contradict one another if the inequality

$$w \ll \sqrt{\frac{e\mathcal{E}_0\kappa}{2m}}$$

is satisfied. It should be emphasized that equation (20) cannot be derived from equation (6) because conditions (10a) and (10b) cannot be satisfied for any finite x, t and $\omega = 0$.

Substituting equation (20) into equation (12), we can find the particle current from the well and, therefore, the ionization probability w per unit of time, because this probability coincides with the electron current from the well. Thus,

$$w \simeq \frac{\mathrm{e}^{\kappa a}}{1+\frac{1}{2}\kappa a}\left(1-\frac{|E_0|}{U_0}\right)2\frac{|E_0|}{\hbar}\mathrm{e}^{-4|E_0|\kappa/3e\mathcal{E}_0}. \tag{22}$$

By considering $a \to 0$, $U_0 \to \infty$, we obtain the ionization probability of a δ-function-like well.

Note that the ionization probability in the low-frequency limit (see equation (19)) differs from that in the dc field of magnitude equal to the ac-field amplitude (see equation (22) for $\mathcal{E}_0 = \mathcal{E}$) by the factor

$$\sqrt{\frac{1}{2\pi\kappa|E_0|}3e\mathcal{E}}.$$

As we have mentioned in section 3, this factor determines the part of the ac-field period where the barrier is minimum and the tunnelling is proceeding most effectively.

Now we are able to consider the ionization process produced by an ac field superimposed on a dc field—i.e. when the total electric field equals $\mathcal{E}_0 + \mathcal{E}\cos\omega t$. We shall confine ourselves to the case where the ac-field frequency is low ($(1/e\mathcal{E}_0)\omega\hbar\kappa \ll 1$) and the amplitude of this field does not exceed the dc field (under the latter condition the particle current runs in one direction). Since the process that we consider is quasistatic, the ionization probability can be calculated by replacing $\mathcal{E}_0$ by $\mathcal{E}_0 + \mathcal{E}\cos\omega t$ in equation (22) and by averaging this expression over the ac-field period. This procedure means that $\omega \gg w$. If the ac-field amplitude obeys the inequality

$$\frac{1}{3e(\mathcal{E}_0+\mathcal{E})^2}2|E_0|\kappa\mathcal{E} \gg 1 \tag{23}$$

then the steepest-descent method [10] can be used in order to calculate the ionization rate averaged over the ac-field period. As a result, we find

$$w \simeq \frac{\mathrm{e}^{\kappa a}}{1+\frac{1}{2}\kappa a}\left(1-\frac{|E_0|}{U_0}\right)2\frac{|E_0|}{\hbar}\exp\left(-\frac{4|E_0|\kappa}{3e(\mathcal{E}_0+\mathcal{E})}\right)\sqrt{\frac{3e(\mathcal{E}_0+\mathcal{E})^2}{8\pi|E_0|\kappa\mathcal{E}}}. \tag{24}$$

We emphasize that under condition (23) the ionization rate in dc and ac fields (see equation (24)) is markedly less than the ionization rate in the dc field $\mathcal{E}_0+\mathcal{E}$. The reason for this difference is as follows. Equation (23) means qualitatively that the ac-field amplitude is sufficiently large and the total electric field changes markedly between values $\mathcal{E}_0-\mathcal{E}$ and $\mathcal{E}_0+\mathcal{E}$. The ionization rate depends sharply on the total field. At those moments when the ac and dc fields are antiparallel, the ionization rate is much less than at the moment when the ac field is a maximum and parallel to the dc field. Therefore, the ionization rate averaged over the ac-field period is essentially less than the ionization rate in the field $\mathcal{E}_0+\mathcal{E}$.

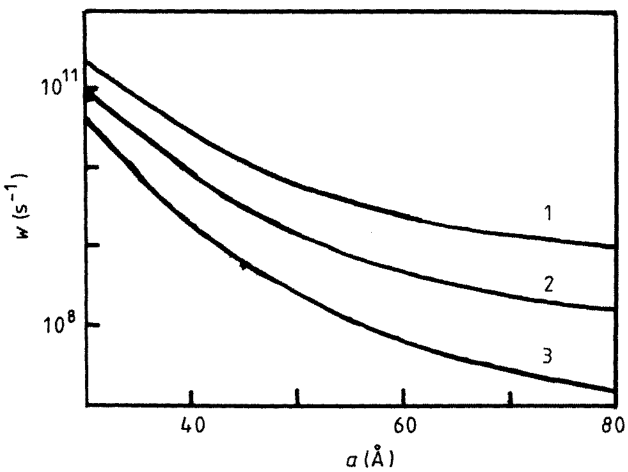

Figure 4. The ionization probability versus the width of a well 0.3 eV deep in the dc electric field of 100 kV cm^{-1} for three amplitudes of an ac low-frequency field: 70 kV cm^{-1} (curve 1), 50 kV cm^{-1} (curve 2), 30 kV cm^{-1} (curve 3).

The dependence of the ionization probability on the well width is represented in figure 4 for one value of the dc field and for three amplitudes of the ac field. The effective electron mass and the well depth are the same as in section 3. As can be seen from figure 4, at fixed values of $\mathcal{E}_0$ and $\mathcal{E}$, the ionization probability decreases monotonically with the increase of the well width. This is connected with the sinking of the energy level and, therefore, with the increase of the effective thickness of the barrier. The ionization probability essentially depends on the ac-field amplitude. The larger the well width is, the stronger this dependence is. Figure 5 shows the dependence of the ionization probability on the ac-electric-field amplitude for two fixed values of the dc field and for a constant width of the well. One can see, in particular, that the ionization probability dependence on the ac-field amplitude becomes less pronounced, if the dc field increases. When obtaining numerical results for figures 4 and 5, we did not verify whether condition (23) was satisfied, and we did not use the approximate equation (24), but replaced $\mathcal{E}_0$ by $\mathcal{E}_0+\mathcal{E}\cos\omega t$ in equation (22) and numerically averaged this expression over the period.

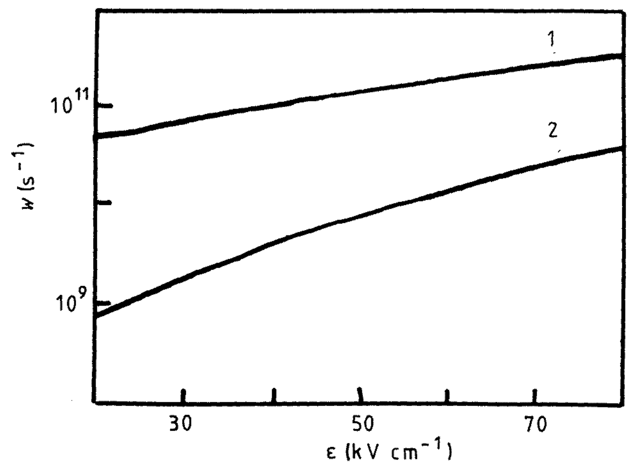

Figure 5. The ionization probability versus the amplitude of a low-frequency ac electric field for a well 0.3 eV deep and 40 Å wide. The dc field equals 150 kV cm^{-1} (curve 1) and 100 kV cm^{-1} (curve 2).

5. Conclusion

We have derived analytical expressions for the multiphoton ionization probability of a rectangular quantum well in an ac electromagnetic field polarized perpendicularly to the well boundaries. The ionization of the ground state, as well as the ionization of higher energy levels, is considered. It has been shown that the qualitative behaviour of the ionization probability as a function of the well width and the ac-field amplitude and frequency is determined by the parameter γ equal to the product of the field frequency and the characteristic time of the electron motion in the classically forbidden region. At high frequencies ($\gamma \gg 1$), the dependence of the ionization probability on the well width, as well as on the field frequency, is essentially nonmonotonic. The peculiarities of this dependence are classified and discussed. In the low-frequency limit ($\gamma \ll 1$), the ionization probability can be treated as the averaged (over the ac-field period) probability of the tunnelling through the potential barrier which is varying in time. A simple analytical expression for this probability has been obtained, and the magnitude of the effect has been estimated.

The problem of the quantum well ionization in an ac low-frequency electric field superimposed on a dc electric field has been solved analytically. The dependence of the ionization probability on the ac-field amplitude and the dc-field magnitude is discussed. It has been shown that this probability decreases monotonically with the increase of the well width.

Appendix A. Solution of the Schrödinger equation

Equation (2) with the initial condition $\Psi(x,0)=\Psi_0(x)$, where $\Psi_0(x)$ is the wave function of the electron stationary state in the quantum well, can be rewritten in the integral form [2]

$$\Psi(x,t)=\int_{-\infty}^{+\infty}G(x,x',t,0)\Psi_0(x')\,\mathrm{d}x' + \frac{1}{\mathrm{i}\hbar}\int_{-\infty}^{+\infty}\mathrm{d}x'\int_0^t\mathrm{d}t'\;G(x,x',t,t')U(x')\Psi(x',t'). \tag{A1}$$

Here $G(x,x',t,t')$ is the Green function defined by the equation

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+e\mathcal{E}x\cos\omega t\right)G(x,x',t,t')=\mathrm{i}\hbar\,\delta(t-t')\,\delta(x-x'). \tag{A2}$$

In the momentum representation, equation (A2) is reduced to that in the first-order partial derivatives. Its exact solution obtained by the method of characteristics is

$$G(x,x',t,t')=\Theta(t-t')\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi\hbar}\;\exp\left\{\frac{\mathrm{i}}{\hbar}[p(t)x-p(t')x']-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\frac{p^2(\tau)}{2m}\,\mathrm{d}\tau\right\} \tag{A3}$$

where $p(t)=p_0+(e\mathcal{E}/\omega)\sin\omega t,\;\Theta(t>0)=1$ and $\Theta(t<0)=0$.

The first term in the right-hand side of equation (A1) describes the quick diffusion of the initial wave function in the absence of the well. As we shall see below, the second term in the right-hand side of equation (A1) gives the value of the electron current running from the well which is not decreasing in time. Thus, only the second term in the right-hand side of equation (A1) will be of interest to us. Assuming the overall loss of carriers to be negligible over the time t of our consideration, i.e. $wt\ll 1$ (w is the ionization probability per unit of time), we suppose

$$\Psi(x',t')\simeq\Psi_0(x')\exp\left(-\frac{\mathrm{i}}{\hbar}E_0t'\right)$$

in the right-hand side of equation (A1). If we take into account expression (3) and fulfil the integration with respect to x' in equation (A1), we obtain for a rectangular well centred at $x=0$

$$\begin{aligned}\Psi(x,t)\simeq{}&\frac{\mathrm{i}U_0}{2\pi\hbar^2}\sqrt{\frac{\kappa}{1+\frac{1}{2}\kappa a}}\int_{-\infty}^{+\infty}\mathrm{d}p_0\;\exp\left[\frac{\mathrm{i}}{\hbar}p(t)x-\frac{\mathrm{i}}{\hbar}\left(\frac{p_0^2}{2m}t+\frac{e^2\mathcal{E}^2}{4m\omega^2}\left(t-\frac{\sin 2\omega t}{2\omega}\right)\right.\right.\\&\left.\left.-\frac{p_0e\mathcal{E}}{m\omega^2}\cos\omega t\right)\right]\int_0^t\mathrm{d}t'\;\exp\left[\frac{\mathrm{i}}{\hbar}\left(|E_0|+\frac{p_0^2}{2m}+\frac{e^2\mathcal{E}^2}{4m\omega^2}\right)t'\right]\\&\times\left\{g(\omega t')\exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{p_0e\mathcal{E}}{m\omega^2}\cos\omega t'+\quad+\frac{e^2\mathcal{E}^2}{8m\omega^3}\sin 2\omega t'\right)\right]\right\}\end{aligned} \tag{A4}$$

where

$$\begin{aligned}g(\omega t')={}&\sin\left(\frac{p_0}{\hbar}+\frac{e\mathcal{E}}{\hbar\omega}\sin\omega t'+k\right)\frac{a}{2}\Big/\left(\frac{p_0}{\hbar}+\frac{e\mathcal{E}}{\hbar\omega}\sin\omega t'+k\right)\\&+\sin\left(\frac{p_0}{\hbar}+\frac{e\mathcal{E}}{\hbar\omega}\sin\omega t'-k\right)\frac{a}{2}\Big/\left(\frac{p_0}{\hbar}+\frac{e\mathcal{E}}{\hbar\omega}\sin\omega t'-k\right).\end{aligned} \tag{A5}$$

The function in the braces in equation (A4) is periodic in t' with a period of $2\pi/\omega$ and can be represented as the Fourier series

$$\sum_{-\infty}^{+\infty}f_n(p_0)\mathrm{e}^{-\mathrm{i}n\omega t'} \tag{A6}$$

with the coefficients defined by the expression

$$f_n(p_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}z\; g(z) \exp\left[\mathrm{i}\left(nz - \frac{p_0 e\mathcal{E}}{\hbar m\omega^2}\cos z - \frac{e^2\mathcal{E}^2}{8\hbar m\omega^3}\sin 2z\right)\right]. \tag{A7}$$

The variable $z = \omega t'$ is introduced here. The coefficients $f_n(p_0 = p_n)$ (the quantity p_n is defined by equation (7a)) are of great importance. We calculate them in appendix B. Let us substitute series (A6) into equation (A4) and integrate the expression obtained with respect to t'. We find

$$\begin{aligned}\Psi(x,t) = \frac{U_0}{2\pi\hbar}\sqrt{\frac{\kappa}{1+\frac{1}{2}\kappa a}}\int_{-\infty}^{\infty}\mathrm{d}p_0 \sum_{n=-\infty}^{\infty} f_n(p_0)\exp\Bigg[\frac{\mathrm{i}}{\hbar}\Bigg[\left(p_0 + \frac{e\mathcal{E}}{\omega}\sin\omega t\right)x \\ + (|E_0| - n\hbar\omega)t + \frac{p_0 e\mathcal{E}}{m\omega^2}\cos\omega t + \frac{e^2\mathcal{E}^2}{8m\omega^3}\sin 2\omega t\Bigg]\Bigg] \\ \times \frac{1}{p_0^2/2m + |E_0| + e^2\mathcal{E}^2/4m\omega^2 - n\hbar\omega} \\ \times\left[1 - \exp\left(-\mathrm{i}\frac{t}{\hbar}\left(\frac{p_0^2}{2m} + |E_0| + \frac{e^2\mathcal{E}^2}{4m\omega^2} - n\hbar\omega\right)\right)\right].\end{aligned} \tag{A8}$$

It can be shown that at large t ($t \gg 2\hbar m/p_n{}^2 \sim 1/\omega$, $t \gg (m/p_n)(x + e\mathcal{E}/m\omega^2) \sim \sqrt{m/2\hbar\omega}(x + e\mathcal{E}/m\omega^2)$) the latter exponential function in equation (A8) oscillates quickly as a function of p_0, does not essentially influence the integral over p_0 and can be omitted. After that the integration can be carried out with the help of the theory of residues. When calculating the integral, one should bear in mind that electrons move only away from the well at large distances. Therefore, for large positive x (see inequalities (10a)) the contribution to integral (A8) is made by the poles p_n, while for large negative x it is by the poles $-p_n$. As a result, we obtain equation (6). Taking into account inequalities (10a), one can see that the restrictions on t, given above, are reduced to conditions (10b).

Appendix B. Calculation of the coefficients of (A7)

The replacement of the variable $\zeta = \sin z$ in equation (A7) enables us to represent the coefficient $f_n(p_n)$ as a contour integral in the complex plane ζ:

$$f_n(p_n) = \frac{1}{2\pi}\oint_C \frac{\mathrm{d}\zeta}{\sqrt{1-\zeta^2}}\;\exp\left(F(\zeta)\right) g_1(\zeta) \tag{B1}$$

where

$$F(\zeta) = \mathrm{i}\left[n\;\arcsin\zeta - \frac{p_n\kappa}{\omega m\gamma}\sqrt{1-\zeta^2} - \frac{\hbar\kappa^2\zeta}{4m\omega\gamma^2}\sqrt{1-\zeta^2}\right] \tag{B2}$$

$$\begin{aligned}g_1(\zeta) = \sin\left(\frac{p_n}{\hbar} + \frac{\kappa}{\gamma}\zeta + k\right)\frac{a}{2}\Big/\left(\frac{p_n}{\hbar} + \frac{\kappa}{\gamma}\zeta + k\right) \\ + \sin\left(\frac{p_n}{\hbar} + \frac{\kappa}{\gamma}\zeta - k\right)\frac{a}{2}\Big/\left(\frac{p_n}{\hbar} + \frac{\kappa}{\gamma}\zeta - k\right).\end{aligned} \tag{B3}$$

The integration contour C surrounds the slit made between the points $\zeta = -1$ and $\zeta = +1$. The function $\sqrt{1-\zeta^2}$ is assumed to be positive at the upper edge of the slit.

The parameter $\hbar\kappa^2/4m\omega\gamma^2$ is the product of $|E_0|/\hbar\omega$ and $1/2\gamma^2$. The first factor is much larger than unity for multiphoton processes. If $\gamma \ll 1$, the function $\mathrm{e}^{F(\zeta)}$ oscillates quickly along the contour C, and we can use the steepest-descent method [10] to calculate the integral (B1). Moreover, it turns out that we can calculate this integral for an arbitrary value of γ in the same way.

Let us assume that the function $g_1(\zeta)$ varies more slowly compared to the function $\mathrm{e}^{F(\zeta)}$. This is possible if the electric field amplitude is not very large, namely, $\mathrm{e}\mathcal{E}a \ll |E_0|$. In this case, the saddle points ζ_0 can be found from the equation $F'(\zeta_0) = 0$, which is equivalent to

$$\zeta_0^2 + 2\frac{p_n\gamma}{\hbar\kappa}\zeta_0 + \frac{2m\omega\gamma^2}{\hbar\kappa^2}n - \frac{1}{2} = 0. \tag{B4}$$

The roots of this equation are

$$\zeta_0^{\pm} = \pm\mathrm{i}\gamma - \frac{p_n\gamma}{\hbar\kappa}. \tag{B5}$$

We transform the integration contour C into a contour which crosses the saddle points. By expanding $F(\zeta)$ in a power series near each saddle point, retaining the terms with powers not larger than two, and using the formulas of the steepest-descent method [10], we obtain

$$f_n(p_n) \simeq \frac{g_1(\zeta_0^+)\mathrm{e}^{F(\zeta_0^+)}}{\sqrt{-2\pi F_0''(\zeta_0^+)}\sqrt{1-(\zeta_0^+)^2}} - \frac{g_1(\zeta_0^-)\mathrm{e}^{F(\zeta_0^-)}}{\sqrt{-2\pi F_0''(\zeta_0^-)}\sqrt{1-(\zeta_0^-)^2}}. \tag{B6}$$

When deriving expression (B6), we implied that the effective intervals of integration in the steepest-descent method

$$|\zeta - \zeta_0^{\pm}| \sim \sqrt{2|F''(\zeta_0^{\pm})|^{-1}}$$

were rather small, and we could omit the terms of order higher than $|\zeta - \zeta_0^{\pm}|^2$. Using the estimate

$$\frac{1}{2!}|F''(\zeta_0^{\pm})(\zeta - \zeta_0^{\pm})^2| \gg \frac{1}{3!}|F'''(\zeta_0^{\pm})(\zeta - \zeta_0^{\pm})^3|$$

and supposing $p_n = 0$ here (the estimates of p_n are given below), we find the condition of validity for equation (B6):

$$\frac{1}{3\gamma}|\zeta - \zeta_0^{\pm}| \sim \frac{1}{3}\sqrt{\frac{\sqrt{1+\gamma^2}}{\gamma}\frac{\hbar\omega}{|E_0|}} \ll 1. \tag{B7}$$

If $\gamma \gtrsim 1$, inequality (B7) evolves as

$$\frac{1}{3}\sqrt{\frac{1}{|E_0|}\hbar\omega} \ll 1$$

and is satisfied due to condition (5) for multiphoton processes. If $\gamma \ll 1$, expression (B7) changes into

$$\frac{1}{3}\sqrt{\frac{1}{|E_0|\gamma}\hbar\omega} = \frac{1}{3}\sqrt{\frac{1}{\kappa|E_0|}e\mathcal{E}} \ll 1.$$

Thus, if $\gamma \ll 1$, condition (5) is insufficient and inequality (11) must be satisfied.

Let us transform equation (B6). As one can see from the final result (equation (8)), the coefficient $f_n(p_n)$ decreases exponentially if p_n increases. If $\gamma \gtrsim 1$, the characteristic values of the ratio $p_n/\hbar\kappa$ (i.e. the values at which $f_n(p_n)$ differs markedly from zero) do not exceed a quantity of the order of $\sqrt{(1/|E_0|)\hbar\omega}$. Obviously, for multiphoton processes we can suppose $p_n/\hbar\kappa \ll 1$. This inequality permits us to expand the exponents $F(\zeta_0^{\pm})$ in power series with respect to p_n, neglecting all the terms of order higher than p_n^2. Besides this, in all factors of equation (B6), except the factors $e^{F(\zeta_0^{\pm})}$, we can consider $p_n = 0$. As a result, we obtain equation (8). If $\gamma \ll 1$, the parameter $p_n\gamma/\hbar\kappa$ appears instead of the parameter $p_n/\hbar\kappa$ which was essential in the previous case. According to the final result (equation (8)), the characteristic values of the ratio $p_n\gamma/\hbar\kappa$ are of the order of

$$\sqrt{\frac{1}{|E_0|\gamma}\hbar\omega} = \sqrt{\frac{1}{\kappa|E_0|}e\mathcal{E}}.$$

Taking into account inequality (11), we can suppose $p_n\gamma/\hbar\kappa \ll 1$ and repeat the same approximate procedure as was used above. Hence, we obtain equation (8) again.

Appendix C. The wave function of an electron, tunnelling from a rectangular potential well in a dc electric field

The wave function of an electron tunnelling from a rectangular potential well subjected to a dc electric field can be found by using the same procedure as in appendix A up to equation (A5), but the limit $\omega \to 0$ should be taken and $\mathcal{E}$ should be replaced by $\mathcal{E}_0$. As a result, instead of equation (A4) we have

$$\Psi(x,t) \simeq \frac{\mathrm{i}U_0}{2\pi\hbar^2}\sqrt{\frac{\kappa}{1+\frac{1}{2}\kappa a}} \int_{-\infty}^{+\infty} \mathrm{d}p_0\ \exp\left(\frac{\mathrm{i}}{\hbar}(p_0+e\mathcal{E}_0 t)x - \frac{\mathrm{i}(p_0+e\mathcal{E}_0 t)^3}{6\hbar m e\mathcal{E}_0}\right) \int_0^t \mathrm{d}t'\ \mathrm{e}^{P(t')}$$
$$\times\left[\frac{\sin\left((p_0+e\mathcal{E}_0 t')/\hbar + k\right)a/2}{(p_0+e\mathcal{E}_0 t')/\hbar + k} + \frac{\sin\left((p_0+e\mathcal{E}_0 t')/\hbar - k\right)a/2}{(p_0+e\mathcal{E}_0 t')/\hbar - k}\right] \quad \text{(C1)}$$

where

$$P(t') = \frac{\mathrm{i}}{6\hbar m e\mathcal{E}_0}(p_0+e\mathcal{E}_0 t')^3 + \frac{\mathrm{i}}{\hbar}|E_0|t'. \quad \text{(C2)}$$

First we calculate the integral over t'. At sufficiently large t (see below), the exponential function in this integral oscillates quickly and we can use the steepest-descent method [10]. If an electric field is not very strong, such that the inequality $e\mathcal{E}_0 a \ll |E_0|$ is satisfied ($e\mathcal{E}_0$ is assumed to be positive), the expression in the square brackets in equation (C1) varies much more slowly compared to the exponential function. In this case, the saddle points in the complex plane t' are determined by the exponential function. They are the roots of the equation $\mathrm{d}P(t')/\mathrm{d}t' = 0$ and can be written as

$$t'_{\pm} = \frac{1}{e\mathcal{E}_0}(-p_0 \pm \mathrm{i}\,\hbar\kappa). \quad \text{(C3)}$$

Since $|\exp P(t'_+)| \ll |\exp P(t'_-)|$, the contour of integration over t' should be deformed so as to cross t'_+ (in accordance with the steepest-descent method). Note that the saddle point t'_+ contributes to the integral if $0 < \mathrm{Re}\, t'_+ < t$. This condition reduces the region of integration over p_0 in equation (C1) to the interval $(-e\mathcal{E}_0 t) < p_0 < 0$. Let us return to the integration over t'. Expanding the function $P(t')$ in a power series near the point t'_+, omitting the term proportional to $(t'-t'_+)^3$ and using the standard formulas [10], we obtain

$$\Psi(x,t) \simeq \frac{\mathrm{i}U_0\sqrt{m}}{\hbar^2\sqrt{2\pi e\mathcal{E}_0}\sqrt{1+\frac{1}{2}\kappa a}}$$
$$\times\left[\frac{\sin(\mathrm{i}\kappa+k)a/2}{\mathrm{i}\kappa+k} + \frac{\sin(\mathrm{i}\kappa-k)a/2}{\mathrm{i}\kappa-k}\right] \mathrm{e}^{-2|E_0|\kappa/3e\mathcal{E}_0} \int_{-e\mathcal{E}_0 t}^{0} \mathrm{d}p_0\ \mathrm{e}^{Q(p_0)} \quad \text{(C4)}$$

where

$$Q(p_0) = \frac{\mathrm{i}}{\hbar}(p_0 + e\mathcal{E}_0 t)x - \frac{\mathrm{i}(p_0 + e\mathcal{E}_0 t)^3}{6\hbar m e\mathcal{E}_0} - \frac{\mathrm{i}\hbar\kappa^2 p_0}{2me\mathcal{E}_0}. \quad \text{(C5)}$$

Expression (C4) is valid, if the next two conditions hold: the effective integration interval

$$|t' - t'_+| \sim \sqrt{2\left|\frac{\mathrm{d}^2 P(t'_+)}{(\mathrm{d}t')^2}\right|^{-1}} = \sqrt{2m(e\mathcal{E}_0\kappa)^{-1}}$$

is essentially less than the total integration interval t, and the effective interval is small enough that we would be able to omit the term proportional to $(t' - t'_+)^3$. These conditions are expressed by inequalities (21*b*).

Let us turn to the integral over p_0 in equation (C4). In order to take it we use the steepest-descent method once again. The saddle points yielded by the equation $\mathrm{d}Q/\mathrm{d}p_0 = 0$ are described by the expression

$$p_0^{\pm} = -e\mathcal{E}_0 t \pm \sqrt{2me\mathcal{E}_0\left(x - \frac{1}{e\mathcal{E}_0}|E_0|\right)}. \quad \text{(C6)}$$

Since the subject of our interest is the wave function determining the electron current at large positive x, we assume $x > (1/e\mathcal{E}_0)|E_0|$. In this case, the saddle points (C6) lie on the real axis of the complex plane p_0, but only p_0^+ belongs to the integration interval. Taking into account that

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}p_0^2}(p_0^+) = -\frac{\mathrm{i}\sqrt{2}}{\hbar\sqrt{me\mathcal{E}_0}}\sqrt{x - \frac{1}{e\mathcal{E}_0}|E_0|}$$

and using the standard technique of the steepest-descent method, for large positive x we find the wave function (20). Analysing the conditions under which the integral over p_0 can be taken by the steepest-descent method (see the paragraph after equation (C5)), we obtain inequality (21*c*).

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